Inertia and Gravitomagnetism

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The Galilei–Newton law of inertia

Textbook wisdom

"A body remains in the state of rest or uniform rectilinear motion, unless this state is changed by the action of external forces".

- Do force-free bodies exist (perhaps only approximately)? If so, how do we recognize them?
- With reference to what spacial reference system?
- With reference to what measure of time?

The Galilei–Newton law of inertia

"Material points of sufficient mutual separation move uniformly along straight lines—*given that one refers the motion to a suitably chosen reference system and that time is suitably defined.*

Who does not feel the embarrassment that lies in such a formulation. But suppressing the final clause means committing a dishonesty".

Albert Einstein, 1920

James Thomson and Peter Guthrie Tait

"A set of points move, Galilei wise, with reference to a system of co-ordinate axes; which may, itself, have any motion whatever. From observation of the *relative* positions of the points, merely, to find such co-ordinate axes."

P. G. Tait, 1884

Tait's Solution: 1

Aim: Reconstruct the inertial system and timescale from an unordered *finite* number of snapshots ("instances") of instantaneous relative spatial configurations.

- Consider $n+1$ mass-points P_i ($0 \leq i \leq n$) moving inertially, i.e. without internal and external forces, in flat space.
- Their trajectories are represented by $n+1$ functions $t \mapsto \vec{x}_i(t)$ with respect to some, yet unspecified, spatial reference frame and timescale.
- However, the only directly measurable quantities at this point are the $n(n+1)/2$ instantaneous mutual separations of the particles, or, equivalently, their squares:

$$R_{ij} := \|\vec{x}_i - \vec{x}_j\|^2 \qquad \text{for} \quad 0 \leq i < j \leq n \,. \tag{1}$$

Tait's Solution: 2

- The knowledge of the $n(n+1)/2$ squared distances, R_{ij}, is equivalent to the $n(n+1)/2$ inner products

$$Q_{ij} := (\vec{x}_i - \vec{x}_0) \cdot (\vec{x}_j - \vec{x}_0) \qquad \text{for} \quad 1 \leq i \leq j \leq n\,. \tag{2}$$

- Their simple linear relations are given by (no summation over repeated indices here)

$$\begin{aligned} R_{ij} &= Q_{ii} + Q_{jj} - 2Q_{ij} && \text{for} \quad 1 \leq i < j \leq n\,, && \text{(3a)} \\ R_{i0} &= Q_{ii} && \text{for} \quad 1 \leq i \leq n\,, && \text{(3b)} \\ Q_{ij} &= \tfrac{1}{2}\big(R_{i0} + R_{j0} - R_{ij}\big) && \text{for} \quad 1 \leq i \leq j \leq n\,. && \text{(3c)} \end{aligned}$$

Tait's Solution: 3

- We seek an inertial system and an inertial timescale, with respect to which all particles move uniformly on straight lines; that is,

$$\vec{x}_i(t) = \vec{a}_i + \vec{v}_i t \qquad \text{for} \quad 0 \leq i \leq n \tag{4}$$

  hold for some *time-independent* vectors $\vec{a}_i$ and $\vec{v}_i$.
- The 11-parameter redundancy by which such inertial systems and timescales are defined, is given by
  a) spatial translations: $\vec{x} \mapsto \vec{x} + \vec{a}$, $\vec{a} \in \mathbb{R}^3$, accounting for three parameters,
  b) three spatial boosts: $\vec{x} \mapsto \vec{x} + \vec{v}t$, $\vec{v} \in \mathbb{R}^3$, accounting for three parameters,
  c) three spatial rotations: $\vec{x} \mapsto \mathbf{R} \cdot \vec{x}$, $\mathbf{R} \in \mathrm{O}(3)$, accounting for three parameters,
  d) time translations: $t \mapsto t + b$, $b \in \mathbb{R}$, accounting for one parameter, and
  e) time dilations: $t \mapsto at$, $a \in \mathbb{R} - \{0\}$, accounting for one parameter.

Tait's Solution: 4

- The redundancies a) and b) are now eliminated by assuming P_0 to rest at the origin of our spatial reference frame. We then have

$$Q_{ij}(t) = \vec{x}_i(t) \cdot \vec{x}_j(t) = \vec{a}_i \cdot \vec{a}_j + t\,(\vec{a}_i \cdot \vec{v}_j + \vec{a}_j \cdot \vec{v}_i) + t^2\,\vec{v}_i \cdot \vec{v}_j\,. \qquad (5)$$

- Measuring the mutual distances, i.e. the Q_{ij}, at k instances t_a ($1 \leq a \leq k$) in time we obtain the $kn(n+1)/2$ numbers $Q_{ij}(t_q)$. From these we wish to determine the following unknowns, which we order in four groups:
  1) the k times t_a,
  2) the $n(n+1)/2$ products $\vec{a}_i \cdot \vec{a}_j$,
  3) the $n(n+1)/2$ products $\vec{v}_i \cdot \vec{v}_j$, and
  4) the $n(n+1)/2$ symmetric products $\vec{a}_i \cdot \vec{v}_j + \vec{a}_j \cdot \vec{v}_i$.

Tait's Solution: 5

- The arbitrariness in choosing the origin and scale of the time parameter t (corresponding to points d) and e) above) can e.g. be eliminated by choosing $t_1 = 0$ and $t_2 = 1$.
- Hence the first group has left the $k - 2$ unknowns $t_3, \dots, t_k$. The last remaining redundancy, corresponding to spatial rotations (point c) above), is *almost* eliminated by choosing P_1 on the z axis and P_2 in the xz plane. This is possible as long as P_0, P_1, P_2 are not colinear. Otherwise we choose three other mass points for which this is true (we exclude the exceptional case where all mass points are colinear). We said that this 'almost' eliminates the remaining redundancy, since a spatial reflection at the origin is still possible.

Tait's Solution: 6

- Tait's strategy is now as follows: for each instant in time t_a consider the $n(n+1)/2$ equations (5). There are $k-2$ unknowns from the first and $n(n+1)/2$ unknowns each from groups 2), 3), and 4). This gives a total of $kn(n+1)/2$ equations for the $k-2+3n(n+1)/2$ unknows.
- The number of equations minus the number of unknowns is

$$(k-3)n(n+1)+2-k\,. \tag{6}$$

- This is positive, if and only if $n \geq 2$ and $k \geq 4$. Hence the minimal procedure is to take four snapshots ($k=4$) of three particles ($n=2$), which results in 12 equations for 11 unknowns.

Ludwig Lange (1863 - 1936)

“a wrongly forgotten person”

Definition An *inertial system* is any reference system with respect to which the trajectories of three force-free mass points, ejected from a common point in three linearly independent directions, are straight lines.

Theorem Relative to an inertial system *any* force-free mass point will move on a straight line.

Definition An *inertial timescale* is any timescale, with respect to which a force-free mass point moves uniformly on its trajectory, i.e. travels equal distances in equal times.

Theorem Relative to an inertial timescale *any* force-free mass point will move uniformly.

L. Lange, 1886

Which astronomical reference system is inertial ?

- ICRS/F (International Celestial Reference System/Frame; 1989): > 600 extragalactic radio sources, located by VLBI (Very Large Baseline Interferometry); Accuracy < 0.1 mas (mas = milli arc-seconds). Presently the best realisation of a “globally non-rotating reference system”.

- HIPPARCOS (HIgh Precision PARalax COllecting Satellite; 1989-93): 120 000 galactic sources (stars); accuracy: < 1.0 mas. Increase by 2 orders of magnitude expected from ESA-Mission GAIA, whose launch is scheduled for 2009.

$\Rightarrow$ galactic rotation: $T = 225 \cdot 10^6$ y $\Rightarrow$ $\Omega = 5.76$ mas/y.

$\Rightarrow$ Can galactic reference-frame be dynamically proven to be non-inertial ?

Newton versus Einstein

NEWTON	EINSTEIN
field: φ (scalar, spin 0)	field: g_{ab} (2. rank tensor, spin 2)
$\Delta\varphi = 4\pi G\,\rho$	$G^{ab}(g, \partial g, \partial^2 g) = \frac{8\pi G}{c^4} T^{ab}$
$\ddot{\vec{x}} = -\vec{\nabla}\varphi$	$\ddot{x}^a + \Gamma^a_{bc}(g, \partial g)\,\dot{x}^b \dot{x}^c = 0$

Newton: Gravitation is a force, i.e. causes deviations from inertial motion. The *global* inertial structure is defined independently (absolute space).

Einstein: Gravitation is identical to inertial structure and hence no force in Newton's sense. Inertial systems are defined *locally* (as field!) any may be relatively accelerated.

GR: T^{ab} as source for the gravitational field

$$T^{ab} = \left(\begin{array}{c|c} W & \frac{1}{c}\,\vec{s}^{\top} \\ \hline c\,\vec{g} & \sigma_{\alpha\beta} \end{array}\right)$$

W : energy density

$\vec{s}$: energy-current density

$\vec{g}$: momentum density

$\sigma_{\alpha\beta}$: momentum-current density ($-$stress tensor)

Formal analogy to electrodynamics

- Let $\eta_{ab} = \mathrm{diag}(-1,1,1,1) =$ Minkowski–metric. Set $g_{ab} = \eta_{ab} + h_{ab}$, where h_{ab} perturbation of the form

$$h_{00} = h_{11} = h_{22} = h_{33} =: \tfrac{2}{c^2}\,\phi \quad \text{and} \quad h_{0\alpha} =: -\tfrac{1}{c}\,A_\alpha$$

- Set $\vec{E} := -\vec{\nabla}\phi$ and $\vec{B} := \vec{\nabla} \times \vec{A}$. Perform linear approximation in field and v/c (weak field low velocities). Einstein's equations become ($\rho := W/c^2$):

$$\Delta\phi = 4\pi G\rho \qquad \Delta\vec{A} = \frac{16\pi G}{c^2}\,\rho\vec{v} \qquad \dot{\vec{v}} = \vec{E} + \vec{v}\times\vec{B}$$

equals static Maxwell eq. for $\varepsilon_0 \to -1/4\pi G$ and $\mu_0 \to -16\pi G/c^2$

Spin precession in magnetic field

- A spinning particle of charge e and mass m is placed in a magnetic field $\vec{B}$. Its spin vector σ will precess according to:

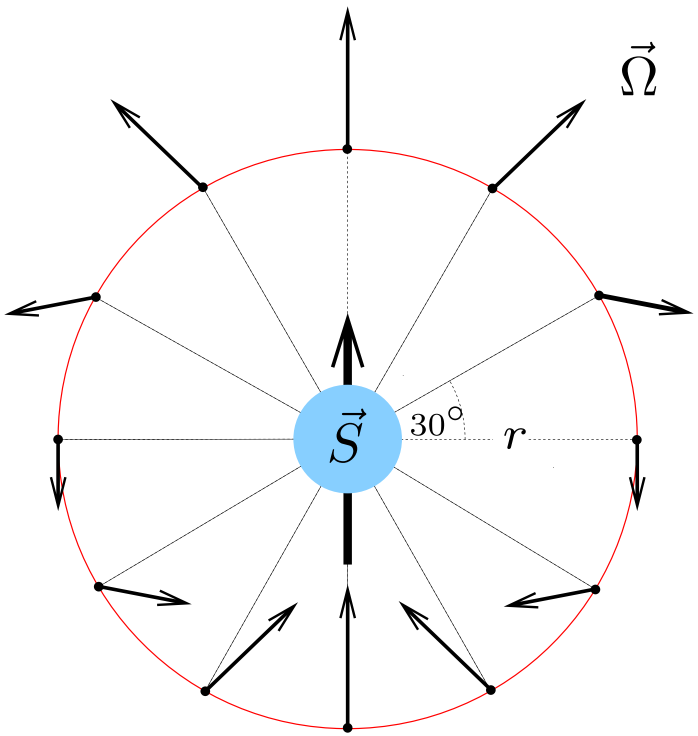
$$\dot{\vec{\sigma}} = \vec{\Omega} \times \vec{\sigma} \qquad \vec{\Omega} = -\frac{g}{2} \cdot \frac{e}{m} \cdot \vec{B}$$

- In the gravitational case have $e = m$ and $g = 1$ (equivalence principle); hence

$$\boxed{\vec{\Omega} = -\tfrac{1}{2}\vec{B}}$$

- The influence of mass-*currents* upon local inertial systems is called *frame dragging* or a *Machian effect*.

Gravitomagnetic dipole



$$\vec{\Omega}(\vec{x}) = \frac{G}{c^2} \cdot \frac{3\vec{n}(\vec{n}\cdot\vec{S}) - \vec{S}}{r^3}$$

at Earth's north-pole (NP):

$$\begin{aligned}\Omega(\text{NP}) &= \frac{4}{5} \cdot \frac{GM/c^2}{R} \cdot \omega \\ &= 5.5 \cdot 10^{-10} \cdot \omega\end{aligned}$$

Effect in polar orbits

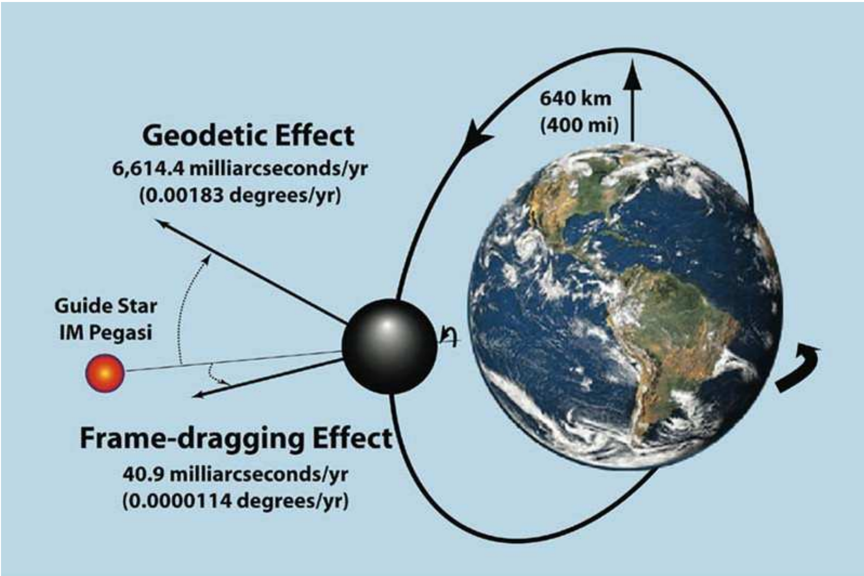
On a polar orbit, the Newtonian effects due to higher multipoles of Earth are suppressed. Averaging $\vec{n}$ along the orbit (i.e. all directions) leads to $\vec{n}(\vec{n} \cdot \vec{S}) \to \vec{S}/2$.

Hence:

$$\vec{\Omega}_{\text{top}}(\text{satellite}) = \frac{1}{5} \cdot \frac{GM/c^2}{(R + 640\,\text{km})^3} \cdot \vec{\omega} \approx 41\,\text{mas/y} \cdot \hat{\omega}$$

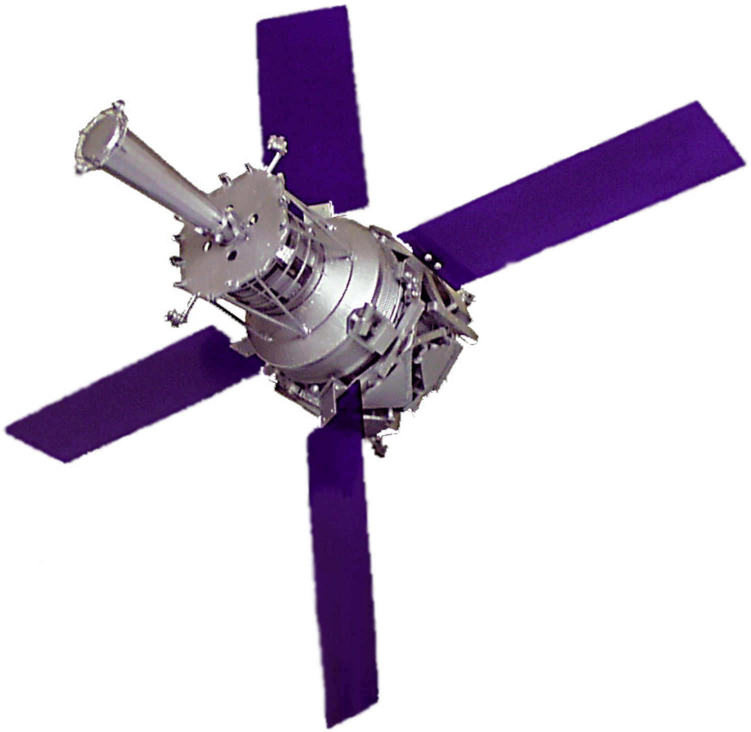
where 'mas'='milli-arc-second' $\approx 3 \cdot 10^{-7}$ angular degrees = opening angle by which an object of size 1.9 m (e.g. an astronaut) on the Moon is seen from Earth!

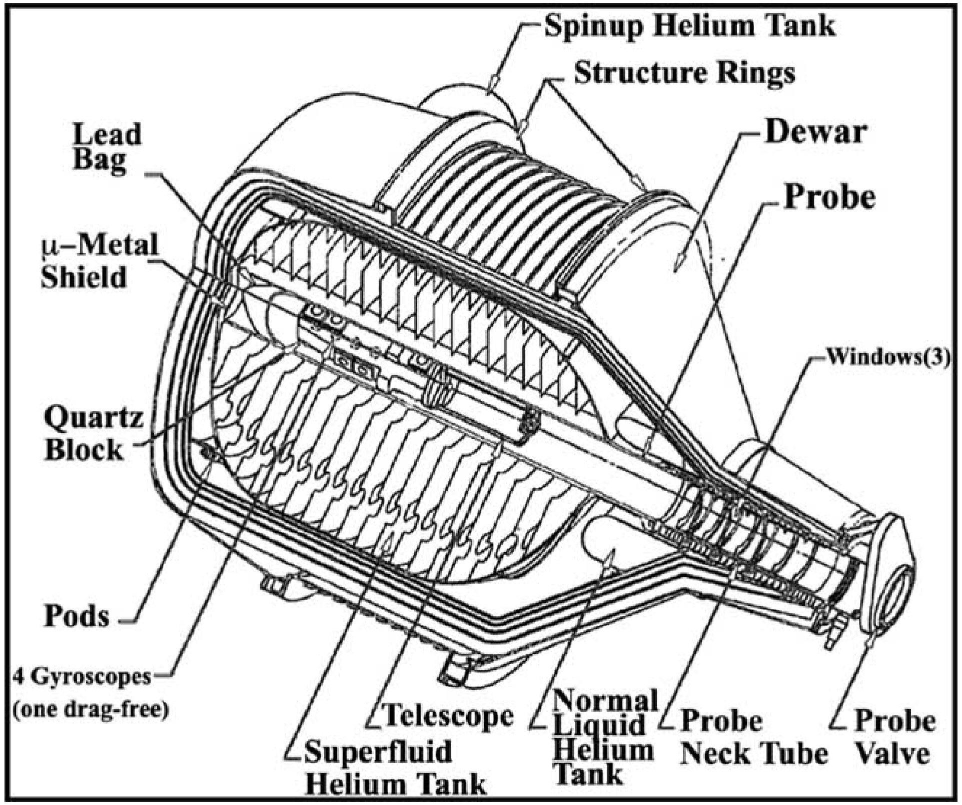
The ‘Gravity Probe-B’ experiment



- Geodetic precession is a consequence of pure spatial curvature (‘electric-’ or ‘Coulomb-part’ of gravitational field): 6.6 as/y
- Lense-Thirring precession is a consequence of mixed space-time curvature (‘magnetic’- oder ‘current-part’ of gravitational field): 0.041 as/y

"On the verge of the technically feasible"





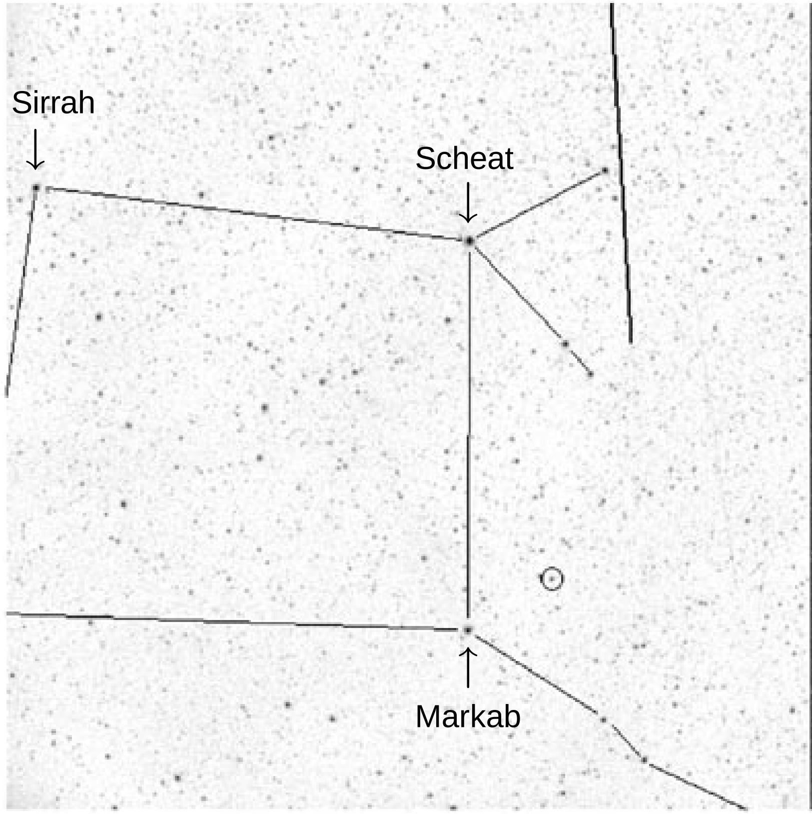
The most perfect man-made spheres

- Niobium-coated balls of fused quartz, $\emptyset = 3.8$ cm, $\nu = 10\,000$ U/min
- Homogeneity = $2 \cdot 10^{-6}$, sphericity = $3 \cdot 10^{-7}$ (40 atom layers $\longrightarrow$ 2 m mountains on earth)
- Orientation control by measurements of magnetic flux using SQUIDs; accuracy 0.1 mas $= 3 \cdot 10^{-8}$ Deg
- Orientation stability $< 3 \cdot 10^{-2}$mas/y
- Temperature $T = 1.82$ K

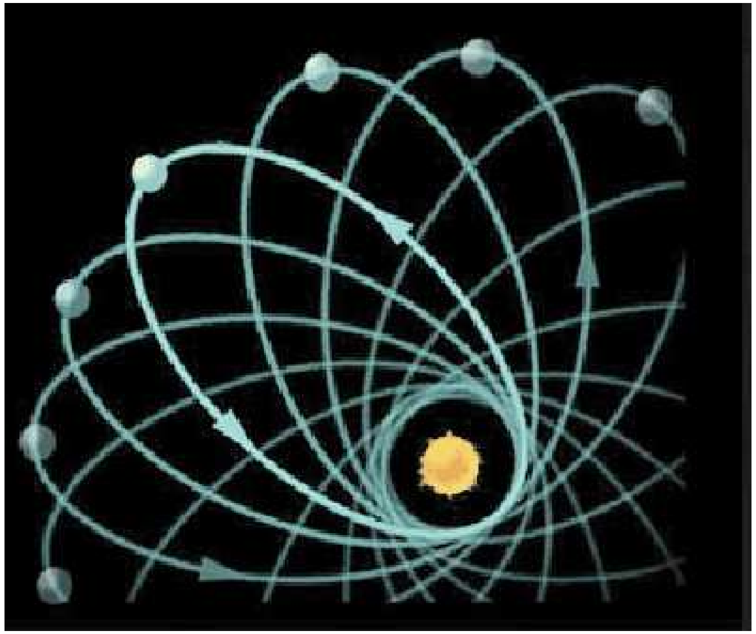
Quasars realise the inertial frame

- Reference direction kept by on-board telescope to star IM Pegasi: red giant, M=$1.5\,M_\odot$, R=$13R_\odot$, magnetically active $\Rightarrow$ radio-loud, $d = 320$ lj
- Proper motion of IM Pegasi: 35 mas/y against quasar-background, monitored by VLBI.
- Long-time accuracy < 0.5 mas/y

Where to find IM Pegasi



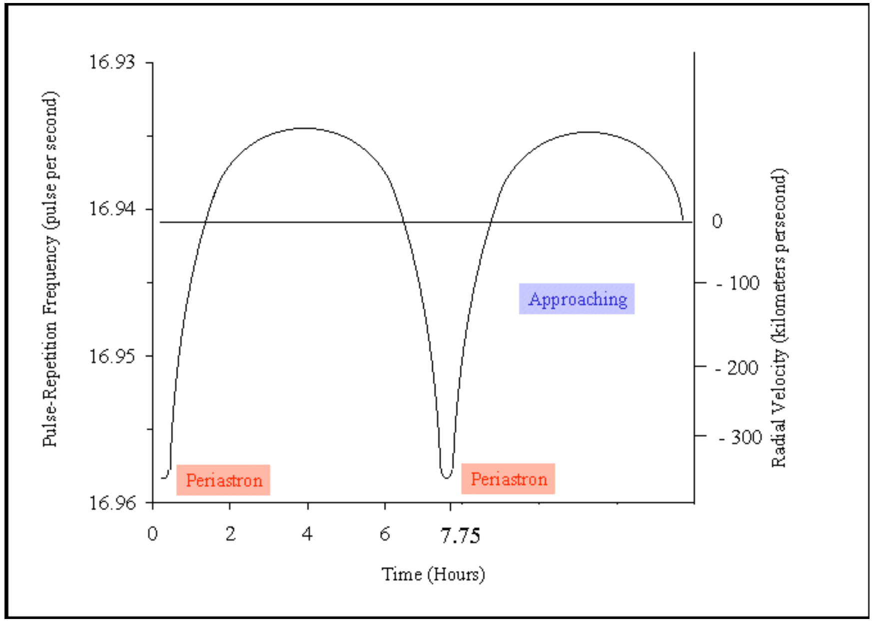
Periastron precession



$$\vec{\Omega} = -\frac{2G}{c^2} \cdot \frac{3\hat{L}(\hat{L} \cdot \vec{S}) - \vec{S}}{a^3(1-\varepsilon^2)^{3/2}}$$

$\hat{L}$ = direction of ang. momentum
a = major semi-axis
ε = orbit's eccentricity

Periastron precession of Hulse-Taylor pulsar

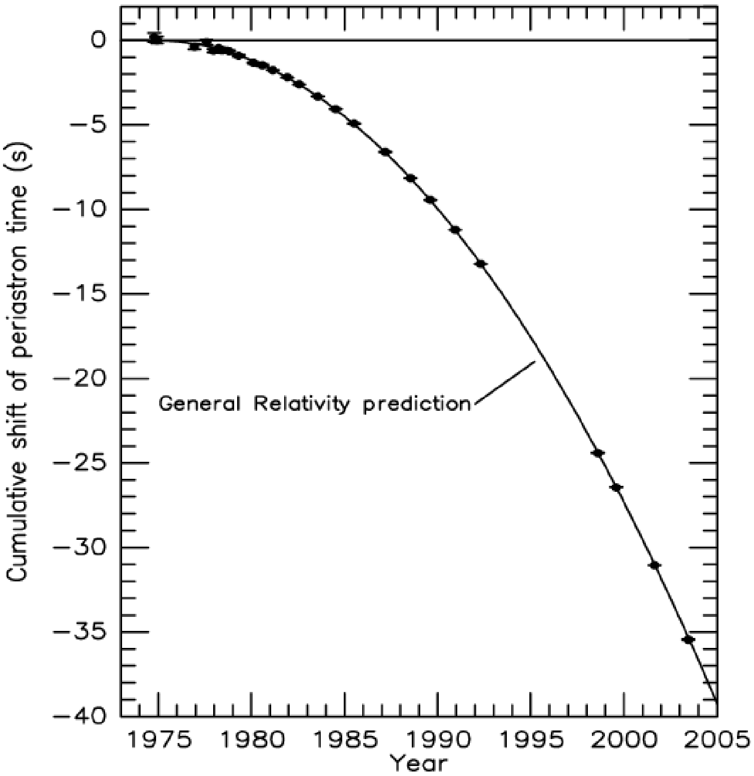


- Measured periastron precession of 4.2 Deg/y in good agreement with GR-prediction.

- Theoretically it is the sum of a 'gravitoelectric' contribution of 10.4 Deg/y and 'gravitomagnetic' contribution, given by -6.3 Deg/y (retrograde!)

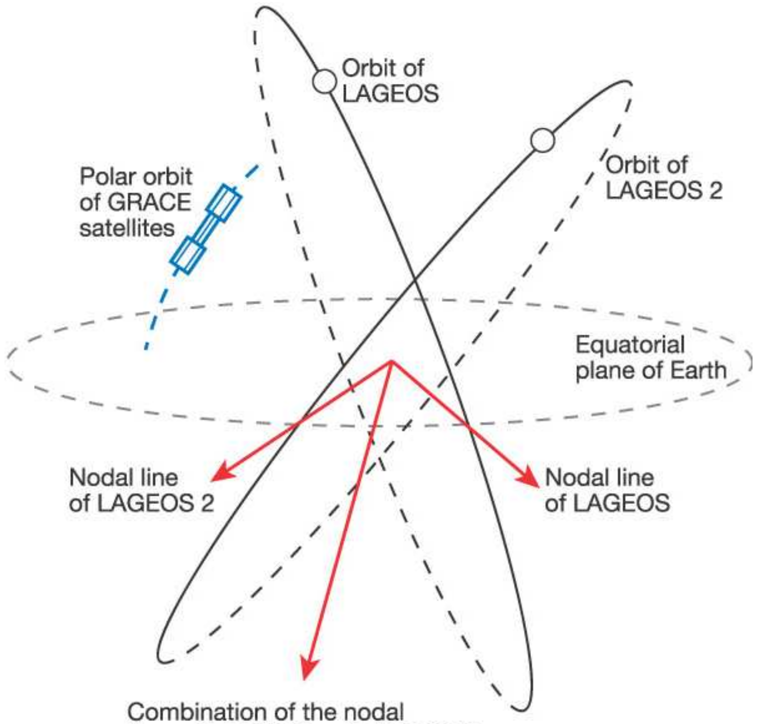
$\Rightarrow$ Establishes *indirect* measurement of gravitomagnetic field.

Decrease of periastron period for the Hulse-Taylor pulsar



gravitational waves !

Precession of orbital planes



$$\vec{\Omega} = \frac{2G}{c^2} \cdot \frac{\vec{S}}{a^3(1-\varepsilon^2)^{3/2}}$$

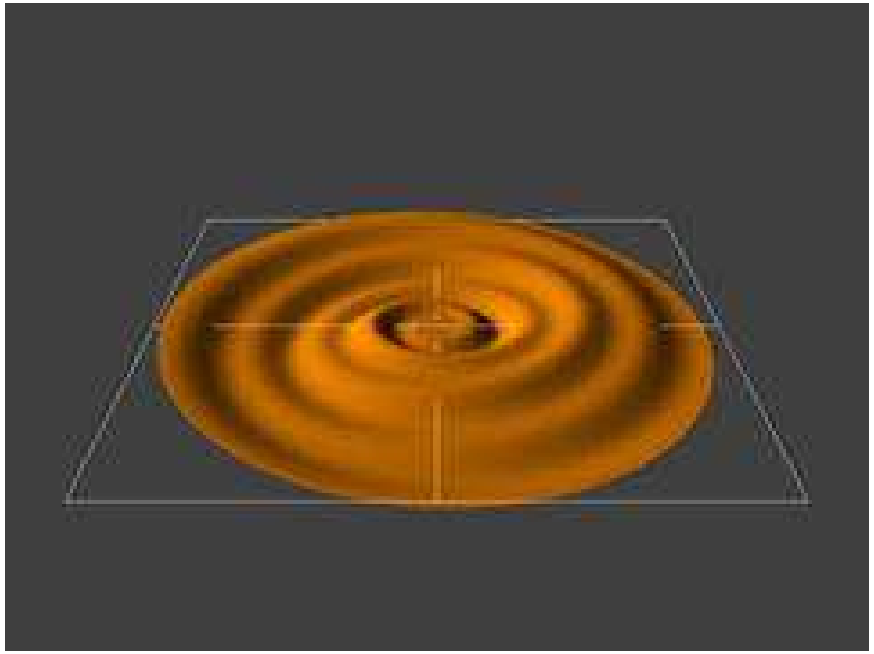
a = semi-major axis

ε = orbit's eccentricity

LAGEOS (LAser GEOdynamic Satellite)

- For orbit around Earth with $a \approx 12\,000\,\text{km}$ and $\varepsilon \approx 0$ have $\Omega = 33\,\text{mas/y}$.
- Confirmed at 10%-level by Ciufolini & Pavlis (Nature, 21. Oct. 2004).
- Main sources of inaccuracy are 1) small orbit-eccentricities (bad localisability of perigee) and 2) current incomplete knowledge of Earth's higher multipole moments. Note: orbits of satellites are neither polar nor supplementary.

Quasi-periodic oscillations in X-ray binaries



- Accretion disc around neutron star. Outer parts of disc 'bulge out' and precess, as consequence of gravitomagnetism. This implies quasi-periodic occlusions of X-ray emission regions and hence of seen X-ray amplitudes. Typical modulation frequencies for few solar mass NS are a few kHz.
- Apparently 'seen' by RXTE (Rossi X-ray Timing Explorer; time resolution: 0.1 ms) in case of 14 systems. (Uncertainties: q-moment, viscosity effects)

The old and new Machian problem

$\Rightarrow$ Mach's Principle (according to A.E.): The totality of all masses and their state of motion determine the local inertial systems.

- In this strict form it cannot be fulfilled in GR, since physical degrees of freedom residing in gravitational field cannot be neglected. (Just as the electromagnetic field is not determined by the charge and current distribution.)
  Replacing 'determine' $\Rightarrow$ 'influence' renders it correct though also more trivial.

- Then the question remains: Why is, apart from local effects of the sort just discussed, the ICRS a globally non-rotating (i.e. inertial) system? Can invoke 'cosmological principles' at a fundamental level, like isotropy. But can this also be understood merely from dynamical reasons? ($\longrightarrow$ boundary conditions). We don't know!

THE END